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WRINKLING OF A CIRCULAR ELASTIC PLATE STAMPED BY A SPHERICAL PUNCH

T. X. Yu† and W. J. Stronge
Department of Engineering, University of Cambridge, Cambridge CB2 1PZ, U.K.

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Abstract—Large deflection theory is used to determine when wrinkling occurs in a simply supported circular elastic plate loaded at the center by a spherical punch. If the edges are free to displace radially, thin plates stamped by intermediate radius punches will develop radial wrinkles near the edge at a central deflection approximately equal to four plate thicknesses, $w_0/h \approx 4$. Initially, there are either three or four wrinkles, but the number increases if the central deflection is increased. With larger radius punches, thin plates do not wrinkle. It is calculated that elastic wrinkling occurs in thin plates ($a/h > 80$) when $a^2/Rh > 8$, where a is the plate radius and R is the punch radius.

INTRODUCTION

A circular plate that is simply supported at the edge is pressed transversely near the center so it stretches and bends to a shallow bowl shape. The in-plane circumferential stress component near the center is tensile, while near the edge, this stress is compressive. As out-of-plane deflection of the center increases, the compressive circumferential stress near the edge increases in both magnitude and extent. When these stresses are sufficiently large and extensive, the equilibrium configuration changes from an axisymmetric bowl to a bowl with radial corrugations near the edge; that is, the edge of the plate buckles. This buckling, which does not reduce the load-carrying capacity of the plate, is called wrinkling.

The elastic wrinkling load and mode shape for a simply supported circular plate will be determined by a method based on energy integrals. The potential energy of the stamped plate is the difference between the strain energy associated with bending and stretching to a deformed configuration and the work done by the applied loads. In this energy method, the deformed configuration is represented by a series of deformation modes which are ideally complete and satisfy the boundary conditions. In practice, series that are truncated or that do not satisfy all the boundary conditions result in approximate solutions. This analysis calculates the displacement field using the term which minimizes the potential energy from a series approximation for the wrinkled shape.

ANALYSIS

In this investigation, an axisymmetric load is applied to the middle of an initially flat, circular plate by a spherical punch, as shown in Fig. 1. The punch initially contacts the flat plate at the centre but as the load increases and the plate deflects, the contact region between punch and plate moves outward[1]. With an elastic plate that is simply supported, there is contact with a spherical punch within a circle of radius b (Fig. 2).

ELASTIC DEFLECTION

If the plate of radius a and thickness h is slowly stamped by a spherical punch of radius R, the portion of plate inside the punch contact circle conforms to the punch and has curvature

$$K_r = K_\theta = \frac{1}{R} = \frac{P}{4\pi D}\left[\frac{(1-\nu)}{2(1+\nu)}\left(1-\frac{b^2}{a^2}\right) - \ln\frac{b}{a}\right] \quad \text{for } r < b. \qquad (1)$$

† Present address: Department of Mechanics, Peking University, Beijing, China.

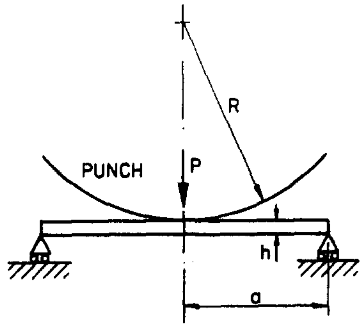

Fig. 1. Punch contacting circular plate.

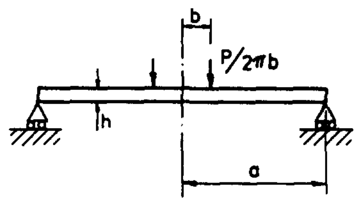

Fig. 2. Contact force distribution for contact radius b.

This curvature results from a constant moment along $r = b$ which is proportional to the total punch force P divided by the flexural rigidity of the plate $D = Eh^3/12(1 - \nu^2)$, where E is Young's modulus and ν is Poisson's ratio for the plate material. The remainder of the plate $r > b$ is a simply supported annulus subjected to the same moment and a transverse line force $P/2\pi b$ along $r = b$.

We have previously shown that the spherical punch pressure on a circular plate may be represented by the union of this pair of edge-loaded plate solutions[1]. Thus, the deflection of an elastic plate stamped by a spherical punch may be shown to be [2, p. 64]

$$w = \begin{cases} \dfrac{P}{8\pi D}\left[(b^2 + r^2)\ln\dfrac{b}{a} + (a^2 - b^2)\dfrac{(3+\nu)a^2 - (1-\nu)r^2}{2(1+\nu)a^2}\right], & \text{for } 0 \le r \le b; \\[2ex] \dfrac{P}{8\pi D}\left\{(a^2 - r^2)\left[1 + \dfrac{(1-\nu)}{2(1+\nu)}\left(1 - \dfrac{b^2}{a^2}\right)\right] + (b^2 + r^2)\ln\dfrac{r}{a}\right\}, & \text{for } b \le r \le a. \end{cases} \tag{2}$$

This transverse deflection may be written in nondimensional form as

$$\delta = \begin{cases} \dfrac{\alpha}{2}\left(\dfrac{A}{B} - \rho^2\right), & \text{for } 0 \le \rho \le \beta; \\[2ex] \dfrac{\alpha}{2B}\left\{\left[(3+\nu) - (1-\nu)\beta^2\right](1 - \rho^2) + 2(1+\nu)(\beta^2 + \rho^2)\ln\rho\right\}, & \text{for } \beta \le \rho \le 1, \end{cases} \tag{3}$$

where

$$\delta = w/h, \qquad \beta = b/a, \qquad \alpha = a^2/Rh$$

$$A = (3+\nu)(1 - \beta^2) + 2(1+\nu)\beta^2\ln\beta$$

$$B = (1-\nu)(1 - \beta^2) - 2(1+\nu)\ln\beta.$$

It follows that the radial inclination is

$$\delta' = \frac{d\delta}{d\rho} = \begin{cases} -\alpha\rho, & \text{for } 0 \le \rho \le \beta; \\ -\alpha\rho[2-(1-\nu)\beta^2-(1+\nu)(\beta^2/\rho^2+2\ln\rho)]/B, & \text{for } \beta \le \rho \le 1. \end{cases} \tag{4}$$

RADIAL DISPLACEMENT AND IN-PLANE FORCES

When elastic deflection of a circular plate is not small in comparison with the plate thickness, the radial displacements cause in-plane forces that are not negligible. If u, w are the radial and transverse deflections, the in-plane forces are

$$\begin{aligned} N_r &= \frac{Eh}{1-\nu^2}\left[\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2+\nu\frac{u}{r}\right] \\ N_\theta &= \frac{Eh}{1-\nu^2}\left[\frac{u}{r}+\nu\frac{du}{dr}+\frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right]. \end{aligned} \tag{5}$$

For equilibrium of forces in the radial direction

$$\frac{d}{dr}(rN_r) - N_\theta = 0.$$

So, for equilibrium,

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2} = -\frac{(1-\nu)}{2r}\left(\frac{dw}{dr}\right)^2-\frac{dw}{dr}\cdot\frac{d^2w}{dr^2}. \tag{6}$$

Nondimensional expressions for eqns (5) and (6) are

$$\begin{aligned} \xi' + \nu\frac{\xi}{\rho} + \frac{h}{2a}\delta'^2 &= a(1-\nu^2)N_r/Eh^2 \\ \nu\xi' + \frac{\xi}{\rho} + \frac{\nu h}{2a}\delta'^2 &= a(1-\nu^2)N_\theta/Eh^2 \end{aligned} \tag{7}$$

and

$$\xi'' + \frac{\xi}{\rho} - \frac{\xi'}{\rho^2} = -\frac{h\delta'}{a}\left(\frac{1-\nu}{2\rho}\delta' + \delta''\right), \tag{8}$$

where the radial displacement $\xi = u/r$.

After substituting the plate inclination (4) into the equilibrium eqn (8), we obtain a differential equation for radial displacement as a function of the radial coordinate

$$\xi'' + \frac{\xi'}{\rho} - \frac{\xi}{\rho^2} = \begin{cases} -h\alpha^2\rho(3-\nu)/2a, & \text{for } 0 \le \rho \le \beta \\ (h\alpha^2/aB^2)[S_1\rho^{-3} + S_2\rho^{-1} + S_3\rho^{-1}\ln\rho + S_4\rho \\ \quad + S_5\rho\ln\rho + S_6\rho(\ln\rho)^2], & \text{for } \beta \le \rho \le 1, \end{cases} \tag{9}$$

where

$$S_1 = (1 + \nu)^3\beta^4/2$$
$$S_2 = -4\nu(1 + \nu)\beta^2 - (1 + \nu)(1 - \nu)^2\beta^4$$
$$S_3 = -2(1 + \nu)^2(1 - \nu)\beta^2$$
$$S_4 = -2(1 - 3\nu) + 4(1 - \nu^2)\beta^2 - (1 - \nu)^2(3 - \nu)\beta^4/2$$
$$S_5 = 8(1 - \nu^2) - 2(1 - \nu^2)(3 - \nu)\beta^2$$
$$S_6 = -2(1 + \nu)^2(3 - \nu).$$

A general solution to this differential equation is determined in the Appendix. This solution is evaluated there for the required boundary conditions when $\nu = 0.3$. If the punch displacement is such that the contact circle radius is half the plate radius, $\beta = 0.5$; distributions of transverse and radial displacement δ, ξ and in-plane forces N_r, N_θ are shown in Fig. 3 for $\nu = 0.3$ and $\beta = 0.5$. At this punch displacement, the radial in-plane force is tensile throughout the entire plate, while the circumferential force N_θ and radial displacement ξ are positive near the center and negative near the edge. The circumferential force changes from tensile to compressive at radius ρ^* somewhat larger than the contact circle radius for $\beta = 0.5$.

The compressive circumferential in-plane force caused by the inward radial displacement of the edge is the source of radial buckles in the outer portion of the plate. The radius of the region where compressive circumferential forces prevail is $\rho > \rho^*$, and this increases with increasing punch deflection, as indicated in Fig. 4. The compressed region is larger than the contact circle for $\delta_0/\alpha < 0.45$ while the die is completely closed at $\delta_0/\alpha = 0.5$, where $\delta_0 \triangleq w_0/h$ is the ratio of center deflection to plate thickness.

WRINKLED CONFIGURATION

Wrinkling occurs when the equilibrium configuration changes from the axisymmetric bowl shape to a bowl containing radial waves in the circumferentially compressed region near the edge. We assume this new configuration is initially a perturbation from the axisymmetric bowl shape and that this perturbation $\bar{\delta}$ is a separable function of radius and azimuth,

$$\bar{\delta} = \Phi(\rho)\psi(\theta). \tag{10}$$

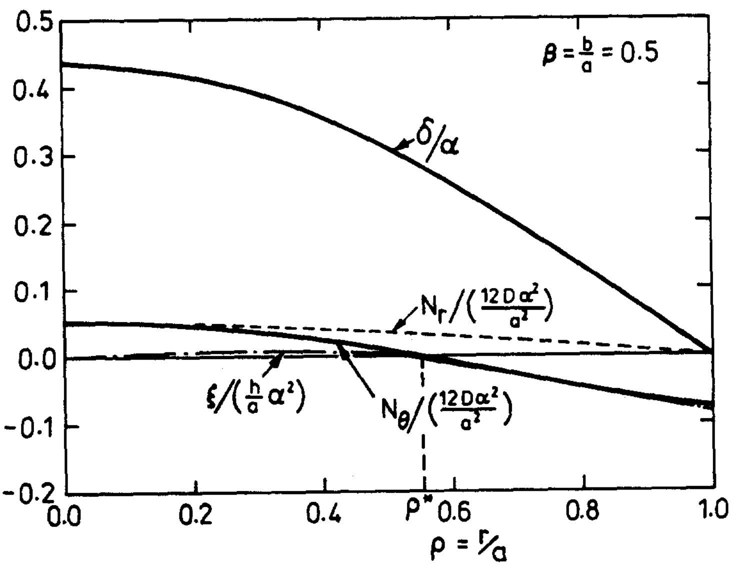

Fig. 3. Transverse and radial displacements and in-plane force components when $b/a = 0.5$.

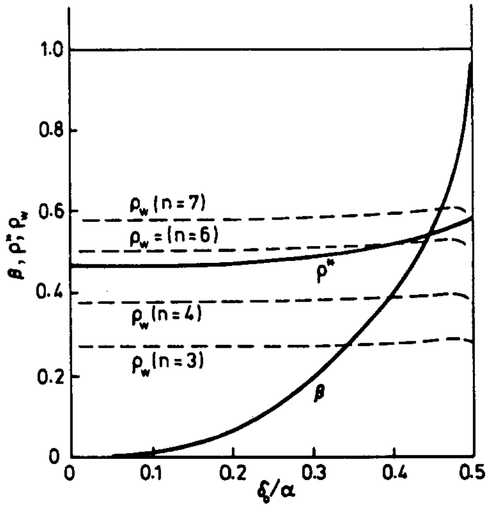

Fig. 4. Comparison of radii for wrinkled and circumferential compressed regions with the contact region.

Because the edge of the plate is free to move away from the support in a direction opposite to the punch displacement

$$\psi(\theta) \le 0 \qquad \text{for } 0 \le \theta \le 2\pi. \tag{11}$$

Thus we assume $\psi(\theta)$ is initially a periodic function of the form

$$\psi(\theta) = \sin(n\theta) - 1, \tag{12}$$

where n is the number of wrinkling waves.

The radial function for wrinkle displacement must satisfy the boundary conditions of zero radial moment at the simply supported edge and compatible displacements at the inner radius of the wrinkled region, ρ_w. Consequently,

$$\tilde{M}_r = -\frac{Dh}{a^2}\left(\frac{\partial^2 \tilde{\delta}}{\partial \rho^2} + \frac{\nu}{\rho}\frac{\partial \tilde{\delta}}{\partial \rho}\right) = 0 \qquad \text{at } \rho = 1. \tag{13}$$

A function that will satisfy this boundary condition is $M_r(\rho) = 0$, which requires

$$\Phi''(\rho) + \frac{\nu}{\rho}\Phi'(\rho) = 0 \qquad \text{for } \rho_w \le \rho \le 1. \tag{14}$$

The solution of this differential equation which satisfies the boundary condition of zero displacements at $\rho = \rho_w$ is $\Phi = (\rho^{1-\nu} - \rho_w^{1-\nu})$. This function does not satisfy the boundary condition of compatible radial inclination at ρ_w, but this is the least important condition in determining the strain energy of the wrinkled configuration. Consequently, we consider

$$\tilde{\delta} = C(\rho_w^{1-\nu} - \rho^{1-\nu})(1 - \sin n\theta) \qquad \text{for } \rho_w \le \rho \le 1, \tag{15}$$

where C is a positive constant.

WRINKLING CRITERION

An energy integral method has been used to establish the wrinkling criterion for circular plates[3]. This method has previously been used to analyse wrinkling of partially clamped circular plates during a deep-drawing process[4].

The bending energy associated with wrinkling will be

$$\Delta U_b = \frac{Dh^2}{2a^2}\int_0^{2\pi}\int_{\rho_w}^{1}\left\{\left(\frac{\partial^2\bar{\delta}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\bar{\delta}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\bar{\delta}}{\partial\theta^2}\right)^2 - 2(1-\nu)\frac{\partial^2\bar{\delta}}{\partial\rho^2}\left(\frac{1}{\rho}\frac{\partial\bar{\delta}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\bar{\delta}}{\partial\theta^2}\right) + 2(1-\nu)\left(\frac{1}{\rho}\frac{\partial^2\bar{\delta}}{\partial\rho\partial\theta} - \frac{1}{\rho^2}\frac{\partial\bar{\delta}}{\partial\theta}\right)^2\right\}\rho\, d\rho\, d\theta. \quad (16)$$

For the wrinkling configuration (15) and with $\nu = 0.3$, the strain energy associated with bending in the wrinkles will be

$$\Delta U_b = \frac{\pi}{2} DC^2 \frac{h^2}{a^2} F(\beta, n, \rho_w), \quad (17)$$

where

$$F(\beta, n, \rho_w) = (0.6282n^4 - 0.8795n^2 + 2.2295)\rho_w^{-0.6} - (1.6667n^4 - 1.9133n^2 + 0.7432) + (1.5385n^4 - 0.3338n^2)\rho_w^{0.7} - (0.5n^4 + 0.7n^2)\rho_w^{1.4}.$$

There is also stretching energy associated with wrinkling at these moderately large displacements. This will be[2]

$$\Delta U_s = \frac{h^2}{2}\int_0^{2\pi}\int_{\rho_w}^{1}\left\{N_r\left(\frac{\partial\bar{\delta}}{\partial\rho}\right)^2 + N_\theta\left(\frac{\partial\bar{\delta}}{\rho\partial\theta}\right)^2\right\}\rho\, d\rho\, d\theta. \quad (18)$$

With the in-plane forces determined in the Appendix (A3) for $\nu = 0.3$,

$$\Delta U_s = 6\pi Dc^2(h^2/a^2)G(\beta, n, \rho_w) \quad (19)$$

where

$$G(\beta, n, \rho_w) = \int_{\rho_w}^{1}\left\{3(1-\nu)^2\rho^{-2\nu}\frac{N_r}{12D\alpha^2/a^2} + n^2(\rho^{-\nu} - \rho_w^{1-\nu}\rho^{-1})^2\frac{N_\theta}{12D\alpha^2/a^2}\right\}\rho\, d\rho. \quad (20)$$

For specified values of the parameters β, n, ρ_w, values of G have been obtained by numerical integration.

When the wrinkling region is outside the contact circle $\rho_w > \beta$, the work done by external force is unchanged by wrinkling. Consequently, wrinkling of the plate initiates when the strain energy associated with bending and stretching in the wrinkle configuration are equal; that is

$$F(\beta, n, \rho_w) = 12\alpha^2 G(\beta, n, \rho_w). \quad (21)$$

The center deflectioin is related to contact circle radius by eqn (3), so

$$\frac{\delta_0}{\alpha} = \frac{A}{2B} = \frac{(3+\nu)(1-\beta^2) + 2(1+\nu)\beta^2 \ln\beta}{2(1-\nu)(1-\beta^2) - 4(1+\nu)\ln\beta}. \quad (22)$$

Thus, the deflection where wrinkling begins is

$$\delta_0 = \frac{A}{2B}\left[\frac{F(\beta, n, \rho_w)}{12G(\beta, n, \rho_w)}\right]^{1/2}. \quad (23)$$

In this criterion, both n and ρ_w are parameters. For each integer value of n, a minimum value of δ_0 which satisfies criterion (23) can be obtained by scanning ρ_w from 0 to 1. The calculated values for inner radius of buckling depend on the wave number as shown in Fig. 4. The inner radius which minimises δ_0 is closer to the center for smaller wave numbers and increases slightly as deflection increases.

ELASTIC LIMITATION

This analysis is limited to elastic states of stress. If the independent moment and in-plane force at yield are $M_e = Yh^2/6$ and $N_e = Yh$ respectively, where Y is the uniaxial yield stress for the plate material, the moments and in-plane forces for the elastic plate must satisfy a yield criterion

$$|m_i| + |f_i| < 1, \qquad i = r, \theta \tag{24}$$

where $m_r = M_r/M_e$, $f_r = N_r/N_e$, and $m_\theta = M_\theta/M_e$, $f_\theta = N_\theta/N_e$.

Yielding first occurs at the center of the plate where

$$\begin{aligned} |m| &= \frac{D(1+\nu)/R}{Yh^2/6} = \frac{E}{Y}\cdot\frac{1}{2(1-\nu)}\frac{h}{R} \\ |f| &= \frac{12D(1+\nu)\alpha^2/a^2}{Yh}\cdot\frac{\overline{C}_{11}}{B^2} = \frac{1}{(1-\nu)}\frac{E}{Y}\frac{a^2}{R^2}\frac{\overline{C}_{11}}{B^2}, \end{aligned} \tag{25}$$

and $\overline{C}_{11}(\beta, \nu)$ is defined in the Appendix. Hence, the yield criterion is

$$\frac{h}{R} + \frac{a^2}{R^2}\cdot\frac{2\overline{C}_{11}}{B^2} < \frac{2(1-\nu)Y}{E}$$

or

$$\frac{h}{a} < \frac{2(1-\nu)Y}{E}\cdot\frac{R}{a} - \frac{a}{R}\cdot\frac{2\overline{C}_{11}}{B^2}. \tag{26}$$

This expression provides an elastic limitation to the plate thickness h/a for a given material and plate geometry, $(1-\nu)Y/E$ and R/a. Since both $\overline{C}_{11}/B^2$ and δ_0/α only depend on the contact circle radius β, eqn (26) also defines an elastic-plastic boundary on the $(1/\alpha, \delta_0)$ map when the material and plate geometry are specified.

DISCUSSION OF NUMERICAL RESULTS

The wrinkling criterion for the smallest wave-numbers n have been computed for $\nu = 0.3$. These punch displacements, where wrinkling initiates, are shown on a $(1/\alpha, \delta_0)$ map, Fig. 5. The region above each curve is where the wrinkled configuration is the equilibrium configuration for this wave number. The curve $\beta = 1$ (i.e., $b = a$) is an upper limit on punch displacement since this represents complete die closure.

Figure 5 indicates that:

(i) Elastic wrinkling does not occur until the deflection of the plate centre is about four times the plate thickness;

(ii) For $\alpha \equiv a^2/Rh > 8$, elastic wrinkling initiates at $\delta_0/h = 4$. For $\alpha < 8$, there is no elastic wrinkling. In thin plates pressed by a large radius punch, there can finally be contact with the punch across the entire surface of the plate without wrinkling;

(iii) Wrinkling initiates in a three or four wave mode of deformation;

(iv) If punch displacement is increased after buckling initiates, the mode of deformation may change to a larger wave number.

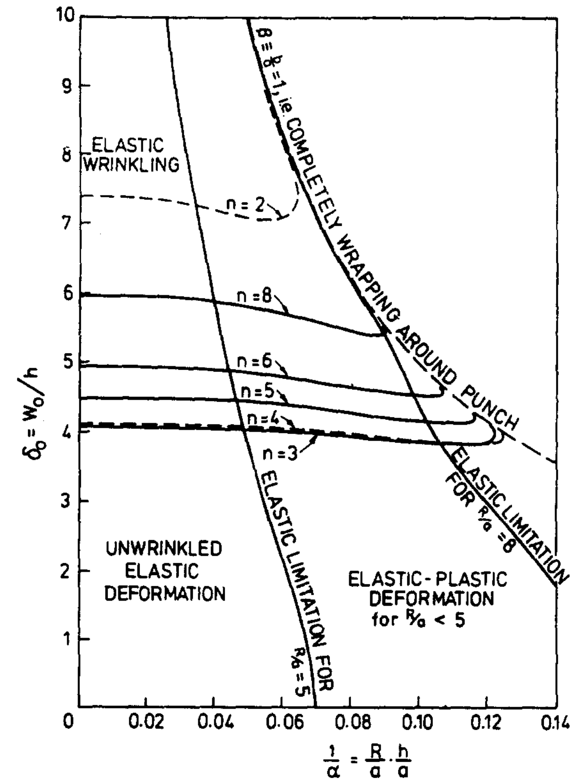

Fig. 5. Boundaries of plate parameters that result in elastic wrinkling.

The punch displacement where plastic deformation begins is also shown on Fig. 5 for $E/Y = 500$ and $R/a = 5$ or 8. Elastic deformations occur with small values of $1/\alpha$ (to the left of the curves) and elastic-plastic deformations occur with larger values of $1/\alpha$. For example, with $E/Y = 500$ and $R/a = 5$, elastic wrinkling develops for $\delta_0 = W_0/h > 4$ if $a/h > 102$. When $a/h < 102$, plastic deformation beginning near the centre precedes elastic wrinkling.

CONCLUSION

The elastic wrinkling of a circular plate loaded by a spherical punch is caused by a compressive circumferential in-plane force that develops at large deflections of the plate. Elastic wrinkling does not occur until $w_0/h \simeq 4$. Then, the wrinkling first occurs in a three or four wave mode. With increasing punch displacement beyond the initiation of wrinkling, the deformation mode of wrinkling changes to a larger number of waves.

Elastic wrinkling develops in thin plates stamped by punches with an intermediate range of punch radius. In thin plates stamped by large radii punches, the plate does not wrinkle before the contact circle between punch and plate has moved out to the edge. In thicker plates ($a/h < 80$), plastic deformation occurs before elastic wrinkling begins. A plastic wrinkling analysis is required for these thick plates.

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APPENDIX

The equilibrium equation has a solution

$$\xi_1 = C_{11}\rho + C_{12}/\rho - h\alpha^2\rho^3(3 - \nu)/16a, \quad \text{for } 0 \le \rho \le \beta$$

$$\xi_2 = C_{21}\rho + C_{22}/\rho + (h\alpha^2/aB^2)\{t_1\rho^{-1}\ln\rho + t_2\rho\ln\rho + t_3\rho(\ln\rho)^2 + t_4\rho^3 + t_5\rho^3\ln\rho + t_6\rho^3(\ln\rho)^2\} \quad \text{for } \beta \le \rho \le 1, \tag{A1}$$

where

$$t_1 = -S_1/2$$
$$t_2 = (2S_2 - S_3)/4$$
$$t_3 = S_3/4$$
$$t_4 = (8S_4 - 6S_5 + 7S_6)/64$$
$$t_5 = (2S_5 - 3S_6)/4$$
$$t_6 = S_6/8.$$

For simplicity, consider $\nu = 0.3$; thus

$$t_1 = -0.549\beta^4$$
$$t_2 = -0.189\beta^2 - 0.319\beta^4$$
$$t_3 = -0.592\beta^2$$
$$t_4 = -1.706 + 0.706\beta^2 - 0.083\beta^4$$
$$t_5 = 2.621 - 0.614\beta^2$$
$$t_6 = -1.141.$$

The constants in the solution may be determined from the boundary conditions (11) and (12). Hence, $C_{12} = 0$, and for $\nu = 0.3$

$$C_{11} = \frac{h}{a}\frac{\alpha^2}{B^2}\{0.776 - 0.403\beta^2 + 2.092\beta^2\ln\beta - 0.593\beta^2(\ln\beta)^2 - 0.412\beta^4 + 0.124\beta^4\ln\beta + 0.040\beta^5\},$$

$$C_{21} = \frac{h}{a}\frac{\alpha^2}{B^2}\{0.776 + 1.902\beta^2 - 1.027\beta^4 + 0.443\beta^4\ln\beta + 0.040\beta^6\} \tag{A2}$$

and

$$C_{22} = \frac{h}{a}\frac{\alpha^2}{B^2}\{-0.682\beta^4 + 0.823\beta^4\ln\beta + 0.074\beta^6\}.$$

Consequently, the in-plane forces in eqn (18) will be

$$N_r = \frac{12D\alpha^2}{a^2B^2}\{1.3\overline{C}_{11} - 0.057B^2\rho^2\},$$
$$N_\theta = \frac{12D\alpha^2}{a^2B^2}\{1.3\overline{C}_{11} - 0.171B^2\rho^2\}, \tag{A3}$$

for $0 \le \rho \le \beta$, and

$$N_r = \frac{12D\alpha^2}{a^2B^2}\Big\{1.3\overline{C}_{21} - 0.7\overline{C}_{22} + (3.3t_4 + t_5)\rho^2 + (3.3t_5 + 2t_6)\rho^2\ln\rho + 3.3t_6\rho^2(\ln\rho)^2 + t_2 + (1.3t_2 + 2t_3)\ln\rho + 1.3t_3(\ln\rho)^2 + t_1\rho^{-2} - 0.7t_1\rho^{-2}\ln\rho + \frac{\rho^2}{2}\Big[-2 + 0.7\beta^2 + 2.6\ln\rho + 1.3\frac{\beta^2}{\rho^2}\Big]^2\Big\},$$

$$N_\theta = \frac{12D\alpha^2}{a^2B^2}\Big\{1.3\overline{C}_{21} + 0.7\overline{C}_{22} + (1.9t_4 + 0.3t_5)\rho^2 + (1.9t_5 + 0.6t_6)\rho^2\ln\rho + 1.9t_6\rho^2(\ln\rho)^2 + 0.3t_2 + (1.3t_2 + 0.6t_3)\ln\rho + 1.3t_3(\ln\rho)^2 + 0.3t_1\rho^{-2} + 0.7t_1\rho^{-2}\ln\rho + 0.15\rho^2\Big[-2 + 0.7\beta^2 + 2.6\ln\rho + 1.3\frac{\beta^2}{\rho^2}\Big]^2\Big\}$$

for $\beta \le \rho \le 1$, with $\overline{C}_{ij} = C_{ij}/(h/a)(\alpha^2/B^2)$.